Lecture4(part2)

Topics covered:
Arithmetic

Multiplication of signed-operands

- Recall we discussed multiplication of unsigned numbers:
  - Combinatorial array multiplier.
  - Sequential multiplier.
- Need an approach that works uniformly with unsigned and signed (positive and negative 2's complement) n-bit operands.
- Booth's algorithm treats positive and negative 2's complement operands uniformly.

Booth's algorithm

- Booth's algorithm applies uniformly to both unsigned and 2's complement signed integers.
  - Basis of other fast product algorithms.
- Fundamental observation:
  - Division of an integer into the sum of block-1's integers.

Suppose we have a 16-bit binary number: 0110011011110110

This number can be represented as the sum of 4 "block-1" integers:

```
 0110000000000000
 0000011000000000
 0000000011110000
+0000000000000110
 0110011011110110
```

Booth's algorithm (contd..)

Suppose Q is a block-1 integer: Q = 0000000001111000 = 120

Then: X.Q = X.120

Now: 120 = 128 - 8, so that X.Q = X.120 = X.(128-8) = X.128 - X.8

And:

Q = 0000000001111000

128 = 0000000010000000

8 = 0000000000001000

If we label the LSB as 0, then the first 1 in the block of 1's is at position 3 and the last one in the block of 1's is at position 6.
As a result:

$$X.Q = X.120 = X.128 - X.8 = X.2^7 - X.2^3$$

Booth's algorithm (contd..)

Representing Block-1 integers

Q is an *n*-bit block-1 unsigned integer:

- -Bit position 0 is LSB.
- -First 1 is in bit position j
- -Last 1 is in bit position k

Then:

$$Q = 2^{k+1} - 2^{j}$$

$$Q.X = X.(2^{k+1} - 2^{j}) = X.\ 2^{k+1} - X.\ 2^{j}$$

Booth's algorithm (contd..)

Let Q be the block-1 integer: $Q = 0111111111111110$

To form the product $X.Q$ using normal multiplication would involve 14 add/shifts (one for each 1-valued bit in multiplier Q).

Since:

$$Q = 2^{15} - 2^{1}$$
$$X.Q = X.(2^{15} - 2^{1})$$

Product X.Q can be computed as follows:

1. Set the Partial Product (PP) to 0.
2. Subtract $X.2^{1}$ from PP.
3. Add $X.2^{15}$ to PP.

Note that $X.2^{j}$ is equivalent to shifting X left j times.

Booth's algorithm (contd..)

If Q is not a block-1 integer, Q can be decomposed so that it can be represented as a sum of block-1 integers.

Suppose: $Q = 0110011011110110$

Q can be decomposed as:

$$
\begin{array}{rl}
0110000000000000 & = 2^{15} - 2^{13} \\
0000011000000000 & = 2^{11} - 2^{9} \\
0000000011110000 & = 2^{7} - 2^{4} \\
+\underline{0000000000000110} & = 2^{3} - 2^{1} \\
0110011011110110 &
\end{array}
$$

Thus,

$$Q.X = X.(2^{15} - 2^{13} + 2^{11} - 2^{9} + 2^{7} - 2^{4} + 2^{3} - 2^{1})$$

Booth's algorithm (contd..)

Inputs: n-bit multiplier Q
n-bit multiplicand x
2n-bit current Partial Product (PP) initially set to 0.
(Upper half of PP is bits n-1 through n)
Q has an added '0' bit attached to the LSB (Q has n+1 bits).

Algorithm: For every bit in Q:

1. Examine the bit and its neighbor to the immediate right.
   If the bit pair is:
   - 00 – do nothing.
   - 01 – Add the multiplicand to the upper half of PP.
   - 10 – Sub the multiplicand from the upper half of PP.
   - 11 – Do nothing.
2. Shift the PP right by one bit, extending the sign bit.

Booth's algorithm (contd..) - Example #1

	01001100	Multiplicand X
	01110111	Multiplier Q
	00000000 00000000	Double-length PP register
Step 0 bit pair 10		
action: SUB	1 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0	
Shift Right	1 1 0 1 1 0 1 0 0 0 0 0 0 0 0 0	
Step 1 bit pair 11		
action: ---	1 1 0 1 1 0 1 0 0 0 0 0 0 0 0 0	
Shift Right	1 1 1 0 1 1 0 1 0 0 0 0 0 0 0 0	
Step 2 bit pair 11		
action: ---	1 1 1 0 1 1 0 1 0 0 0 0 0 0 0 0	
Shift Right	1 1 1 1 0 1 1 0 1 0 0 0 0 0 0 0	
Step 3 bit pair 01		
action: ADD	0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 0	
Shift Right	0 0 1 0 0 0 0 1 0 1 0 0 0 0 0 0	
Step 4 bit pair 10		
action: SUB	1 1 0 1 0 1 0 1 0 1 0 0 0 0 0 0	
Shift Right	1 1 1 0 1 0 1 0 1 0 1 0 0 0 0 0	
Step 5 bit pair 11		
action: ---	1 1 1 0 1 0 1 0 1 0 1 0 0 0 0 0	
Shift Right	1 1 1 1 0 1 0 1 0 1 0 1 0 0 0 0	
Step 6 bit pair 11		
action: ---	1 1 1 1 0 1 0 1 0 1 0 1 0 0 0 0	
Shift Right	1 1 1 1 1 0 1 0 1 0 1 0 1 0 0 0	
Step 7 bit pair 01		
action: ADD	0 1 0 0 0 1 1 0 1 0 1 0 1 0 0 0	
Shift Right	0 0 1 0 0 0 1 1 0 1 0 1 0 1 0 0	= Product: 9044

Booth's algorithm (contd..) - Example #2

Multiplicand in Binary:	1 0 1 1 0 1 0 0 = -119
Multiplier in Binary:	0 1 1 1 0 1 1 1 = 76
Step 0 bit pair 10	
action: SUB	0 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0
Shift Right	0 0 1 0 0 1 1 0 0 0 0 0 0 0 0 0
Step 1 bit pair 11	
action: ---	0 0 1 0 0 1 1 0 0 0 0 0 0 0 0 0
Shift Right	0 0 0 1 0 0 1 1 0 0 0 0 0 0 0 0
Step 2 bit pair 11	
action: ---	0 0 0 1 0 0 1 1 0 0 0 0 0 0 0 0
Shift Right	0 0 0 0 1 0 0 1 1 0 0 0 0 0 0 0
Step 3 bit pair 01	
action: ADD	1 0 1 1 1 1 0 1 1 0 0 0 0 0 0 0
Shift Right	1 1 0 1 1 1 1 0 1 1 0 0 0 0 0 0
Step 4 bit pair 10	
action: SUB	0 0 1 0 1 0 1 0 1 1 0 0 0 0 0 0
Shift Right	0 0 0 1 0 1 0 1 0 1 1 0 0 0 0 0
Step 5 bit pair 11	
action: ---	0 0 0 1 0 1 0 1 0 1 1 0 0 0 0 0
Shift Right	0 0 0 0 1 0 1 0 1 0 1 1 0 0 0 0
Step 6 bit pair 11	
action: ---	0 0 0 0 1 0 1 0 1 0 1 1 0 0 0 0
Shift Right	0 0 0 0 0 1 0 1 0 1 0 1 1 0 0 0
Step 7 bit pair 01	
action: ADD	1 0 1 1 1 0 0 1 0 1 0 1 1 0 0 0
Shift Right	1 1 0 1 1 1 0 0 1 0 1 0 1 1 0 0 = Product: -9044

Multiplier: Positive
Multiplicand: 2's comp. -ve

Booth's algorithm (contd..) – Example #3

Multiplicand in Binary:	1 0 1 1 0 1 0 0 = -119
Multiplier in Binary:	1 0 0 0 1 0 0 1= -76
Step 0 bit pair 10	
action: SUB	0 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0
Shift Right	0 0 1 0 0 1 1 0 0 0 0 0 0 0 0 0
Step 1 bit pair 01	
action: ADD	1 1 0 1 1 0 1 0 0 0 0 0 0 0 0 0
Shift Right	1 1 1 0 1 1 0 1 0 0 0 0 0 0 0 0
Step 2 bit pair 00	
action: ---	1 1 1 0 1 1 0 1 0 0 0 0 0 0 0 0
Shift Right	1 1 1 1 0 1 1 0 1 0 0 0 0 0 0 0
Step 3 bit pair 10	
action: SUB	0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 0
Shift Right	0 0 1 0 0 0 0 1 0 1 0 0 0 0 0 0
Step 4 bit pair 01	
action: ADD	1 1 0 1 0 1 0 1 0 1 0 0 0 0 0 0
Shift Right	1 1 1 0 1 0 1 0 1 0 1 0 0 0 0 0
Step 5 bit pair 00	
action: ---	1 1 1 0 1 0 1 0 1 0 1 0 0 0 0 0
Shift Right	1 1 1 1 0 1 0 1 0 1 0 1 0 0 0 0
Step 6 bit pair 00	
action: ---	1 1 1 1 0 1 0 1 0 1 0 1 0 0 0 0
Shift Right	1 1 1 1 1 0 1 0 1 0 1 0 1 0 0 0
Step 7 bit pair 10	
action: SUB	0 1 0 0 0 1 1 0 1 0 1 0 1 0 0 0
Shift Right	0 0 1 0 0 0 1 1 0 1 0 1 0 1 0 0 = Product: 9044

Multiplier:
2's comp. -ve
Multiplicand:
2's comp. -ve

Booth's algorithm and signed integers

Booth's algorithm works correctly for any combination of 2's complement numbers.
If Q is a 2's complement negative number, the Q is of the form:

$$Q = 1\ q_{n-2}\ q_{n-3}\ q_{n-4} \ldots\ldots\ldots\ q_1\ q_0$$

And:

$$V(Q) = -2^{n-1} + q_{n-2}\, 2^{n-2} + q_{n-3}\, 2^{n-3} + \ldots\ldots\ldots + q_2\, 2^2 + q_1\, 2^1 + q_0\, 2^0$$

If Q has no zeros, that is: $Q = 111....11 = -1$

Booth technique will:

- See the first pair as 10 (subtract the multiplicand)
- See all other pairs as 11 (do nothing)

Thus, Booth technique will compute the result as required:

$$0 - X = -X = -1.X$$

Booth's algorithm and signed integers (contd..)

Booth's technique is correct for an arbitrary –ve 2's complement number Q is a 2's complement number:

$$Q = 1\, q_{n-2}\, q_{n-3}\, q_{n-4} \ldots\ldots\ldots\ldots\, q_1\, q_0$$

$$V(Q) = -2^{n-1} + q_{n-2}\, 2^{n-2} + q_{n-3}\, 2^{n-3} + \ldots\ldots\ldots + q_2\, 2^2 + q_1\, 2^1 + q_0\, 2^0$$

If we read Q from MSB (left hand side), Q will have certain number of 1s followed by a 0. If the first 0 appears in the m^{th} bit of Q.

$$V(Q) = -2^{n-1} + q_{n-2}\, 2^{n-2} + q_{n-3}\, 2^{n-3} + \ldots\ldots\ldots + q_2\, 2^2 + q_1\, 2^1 + q_0\, 2^0$$

$$= (-2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots\ldots\ldots\, 2^{m+1}) + (q_{m-1}\, 2^{m-1} + \ldots\ldots\ldots + q_2\, 2^2 + q_1\, 2^1 + q_0\, 2^0)$$

$$= -2^{m+1} + (q_{m-1}\, 2^{m-1} + \ldots\ldots\ldots + q_2\, 2^2 + q_1\, 2^1 + q_0\, 2^0)$$

Where $(-2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots\ldots\ldots\, 2^{m+1}) = -2^{m+1}$

- Bit m is 0, so m-bit number starting from 0 is a positive number. Booth's algorithm will work correctly.
- Transitioning from bit m to $m+1$, algo sees the bit pair 10, causing it to subtract the multiplicand 2^{m+1} from the PP.
- All other transition see pairs 11 which have no effect.

Unsigned division

- Division is a more tedious process than multiplication.
- For the unsigned case, there are two standard approaches:
  1.) Restoring division. 2.) Non restoring division.

```
         21
    ________
 13 ) 274
      26
    ____
       14
       13
     ____
        1
```

Try dividing 13 into 2.
Try dividing 13 into 26.

```
               10101
       _____________
 1101 ) 10001001
        01101
        _____
          10000
           1101
          _____
            1110
            1101
           _____
               1
```

Try dividing 1101 into 1, 10, 100, 1000 and 10001.

Restoring division

How do we know when the divisor has gone into part of the dividend correctly?

```
          10101
      ___________
1101 ) 10001001
       01101
       -----
         10000
          1101
         -----
           1110
           1101
           ----
              1
```

Subtract 1101 from 1, result is negative
Subtract 1101 from 10, result is negative.
Subtract 1101 from 100, result is negative
Subtract 1101 from 1000, result is negative.
Subtract 1101 from 10001, result is positive.

Restoring division

Strategy for unsigned division:

Shift the dividend one bit at a time starting from MSB into a register.

Subtract the divisor from this register.

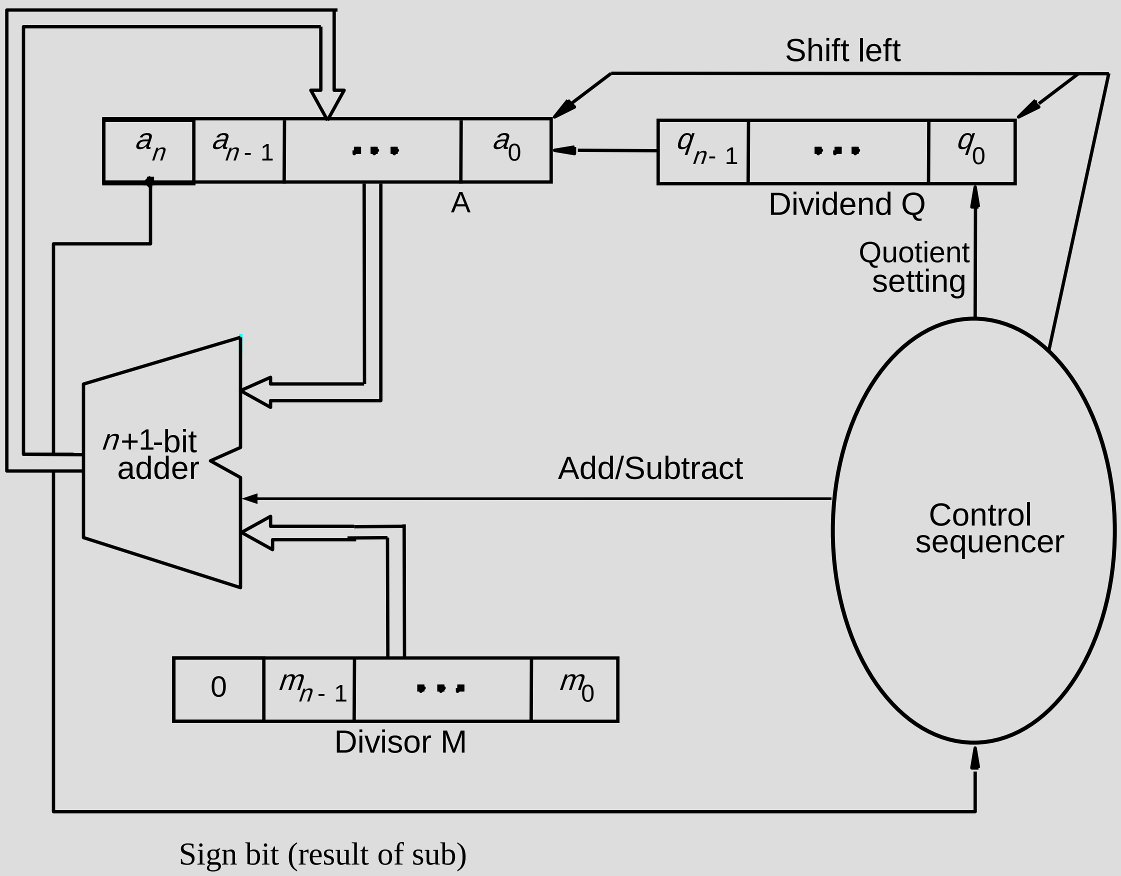
If the result is negative ("didn't go"):

- Add the divisor back into the register.
- Record 0 into the result register.

If the result is positive:

- Do not restore the intermediate result.
- Set a 1 into the result register.

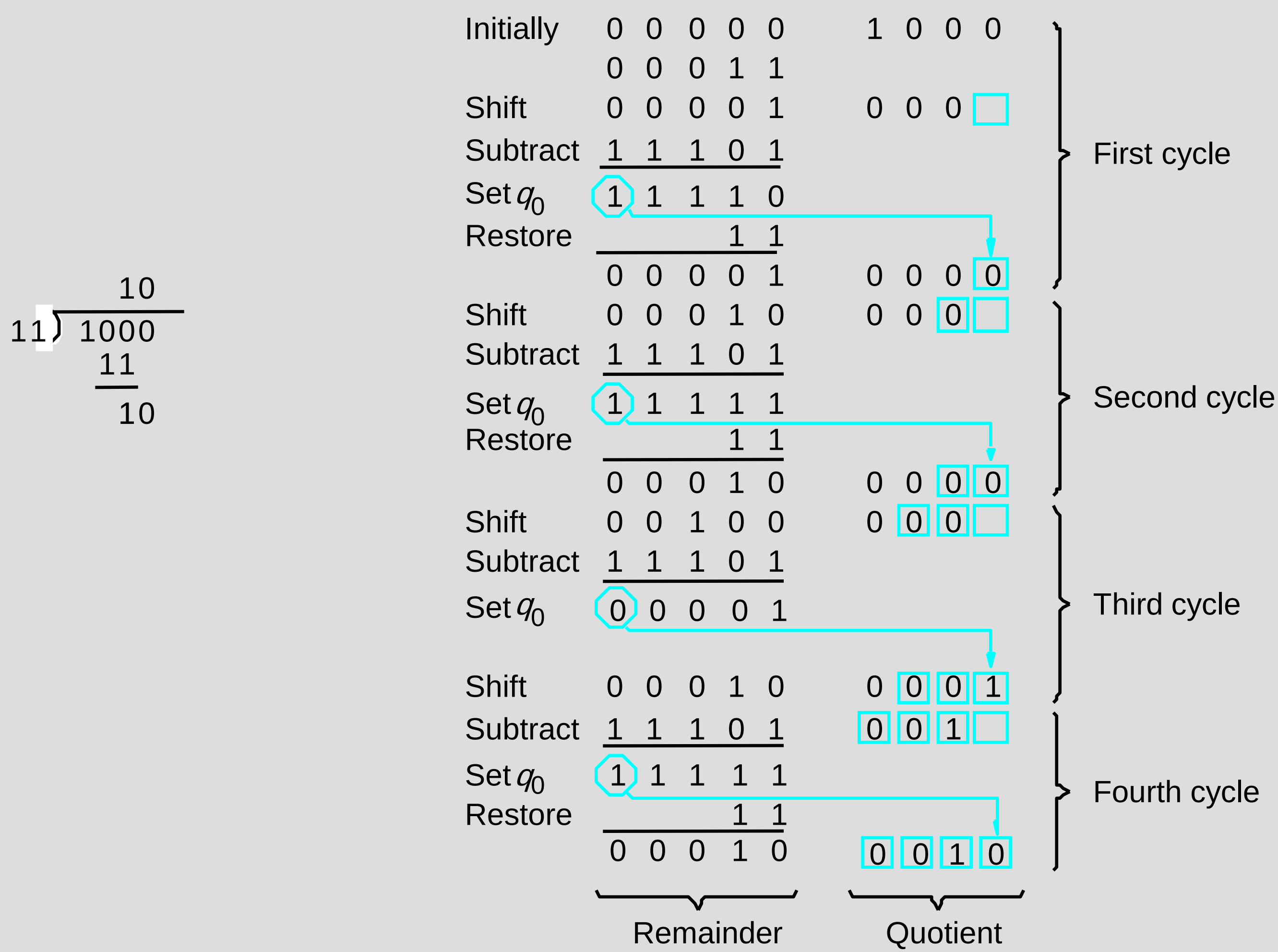
Restoring division (contd..)



Set Register A to 0.
Load dividend in Q.
Load divisor into M.
Repeat n times:
- *Shift A and Q left one bit.*
- *Subtract M from A.*
- *Place the result in A.*
- *If sign of A is 1, set q_0 to 0 and add M back to A. Else set q_0 to 1.*

End of the process:
- *Quotient will be in Q.*
- *Remainder will be in A.*

Restoring division (contd..)



Non-restoring division

Restoring division can be improved using non-restoring algorithm

The effect of restoring algorithm actually is:
If A is positive, we shift it left and subtract M, that is compute $2A\text{-}M$
If A is negative, we restore it ($A+M$), shift it left, and subtract M, that is, $2(A+M) - M = 2A+M$.
Set q_0 to 1 or 0 appropriately.

Non-restoring algorithm is:
Set A to 0.
Repeat n times:
If the sign of A is positive:
Shift A and Q left and subtract M. Set q0 to 1.
Else if the sign of A is negative:
Shift A and Q left and add M. Set q0 to 0.
If the sign of A is 1, add A to M.

Non-restoring division (contd..)

```
      10
11 ) 1000
      11
      ---
       10
```

	A	Q	
Initially	0 0 0 0 0	1 0 0 0	First cycle
	0 0 0 1 1		
Shift	0 0 0 0 1	0 0 0	
Subtract	1 1 1 0 1		
Set q_0	1 1 1 1 0	0 0 0 0	
Shift	1 1 1 0 0	0 0 0	Second cycle
Add	0 0 0 1 1		
Set q_0	1 1 1 1 1	0 0 0 0	
Shift	1 1 1 1 0	0 0 0	Third cycle
Add	0 0 0 1 1		
Set q_0	0 0 0 0 1	0 0 0 1	
Shift	0 0 0 1 0	0 0 1	Fourth cycle
Subtract	1 1 1 0 1		
Set q_0	1 1 1 1 1	0 0 1 0	
		Quotient	

	1 1 1 1 1	Restore remainder
Add	0 0 0 1 1	
	0 0 0 1 0	
	Remainder	